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1993 J. Phys. A: Math. Gen. 26 1775

(http://iopscience.iop.org/0305-4470/26/8/007)

J. Phys. A: Math. Gen. 26 (1993) 1775-1782. Printed in the UK

On a specific interaction of quantum particles

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Received 24 February 1992, in final form 12 August 1992

Abstract. The quantum problem of two particles in a small region of dimensions comparable to those of an existing background interaction is examined. To this purpose, one uses the result that quantum phenomena in the Euclidean formulation of the theory are due to a stochastic spacetime background interaction, whose essence is the time derivative of the Wiener process. The problems of calculating both the transition probability and the path integral for that system are then solved. The specific interaction taking place in this case is likely to play an important role in the quantum description of nucleons.

It is well known (Kač 1957, Migdal 1986, Beilinson 1964, Nelson 1964) that there is a close relationship between Brownian motion and quantum mechanics. In this sense the solution of the time-dependent Schrödinger equation

$$i\hbar\frac{\partial\psi(x,t)}{\partial t}=\hat{H}\psi(x,t) \tag{1}$$

can be obtained from Bloch equation

$$\frac{\partial Z(x,t)}{\partial t}=-\hat{H}Z(x,t) \tag{2}$$

through analytic continuation of $Z(x,t)$, relative to variable t, up to the imaginary axis. Formally, it means the substitution of t by it and thus one gets the transition $Z(x,it)=\psi(x,t)$.

Let us consider a system of Langevine stochastic equations

$$\dot{x}_j-\frac{\partial S(x,t)}{\partial x_j}=\dot{\varphi}_j \qquad j=1,2,\ldots,n \tag{3}$$

where φ_j is the jth component of Wiener's n-dimensional process. The Fokker–Planck equation for this system has the form

$$\frac{\partial W}{\partial t}+\frac{\partial}{\partial x_j}\left(\frac{\partial S}{\partial x_j}W\right)=\frac{1}{4}\frac{\partial^2 W}{\partial x_j^2}. \tag{4}$$

The fundamental solution of this equation, the transition probability density $W(x_0,0;x,t)$, can be obtained from (3) by substitution of variables (Gelfand 1961, Migdal 1986) in Wiener's integral, as

$$W(x_0, 0; x, t) = \int_{c_1} \cdots \int_{c_n} \exp\left[-\int_0^t \left(\dot{x}_j \frac{\partial S}{\partial x_j}\right)^2 \mathrm{d}\tau\right] \exp\left(-\frac{1}{2}\int_0^t \frac{\partial^2 S}{\partial x_j^2} \mathrm{d}\tau\right) \times \prod_{\tau=0}^{t} \frac{\mathrm{d}x_1(\tau)\ldots \mathrm{d}x_n(\tau)}{(\sqrt{\pi\,\mathrm{d}\tau})^n} \tag{5}$$

where the integrals are calculated with respect to all the continuum paths with fixed extremities. Separating the total derivative $\mathrm{d}S/\mathrm{d}t$ in (5), one obtains the so-called factorization theorem

$$W(x_0, 0; x, t) = \frac{\mathrm{e}^{2S(x,t)}}{\mathrm{e}^{2S(x_0,0)}} Z(x_0, 0; x, t) \tag{6}$$

where

$$Z(x_0, 0; x, t) = \int_c \exp\left\{-\int_0^t \left[\left(\frac{\partial S}{\partial x_j}\right)^2 + \frac{1}{2}\frac{\partial^2 S}{\partial x_j^2} + 2\frac{\partial S}{\partial \tau}\right] \mathrm{d}\tau\right\} \mathrm{d}_w x(\tau). \tag{7}$$

In (7),

$$\mathrm{d}_w x(\tau) = \exp\left[-\int_0^t \dot{x}^2(\tau)\,\mathrm{d}\tau\right] \prod_{\tau=0}^{t} \frac{\mathrm{d}x(\tau)}{(\sqrt{\pi\,\mathrm{d}\tau})^n} \tag{8}$$

is the Wiener measure (Gelfand and Vilenkin 1961) and $\mathrm{d}x(\tau) = \mathrm{d}x_1(\tau)\ldots\mathrm{d}x_n(\tau)$. Performing the analytic continuation with respect to time t, $Z(x_0, 0; x, t)$ becomes the solution of the Schrödinger equation for the following Hamiltonian:

$$\hat{H} = -\frac{1}{4}\frac{\partial^2}{\partial x^2} + V(x). \tag{9}$$

In this case (7) must be considered as the Feynman's path integral expression for that solution. Equation (7) is Kač's formula for a potential energy

$$V(x, t) = \left(\frac{\partial S}{\partial x_j}\right)^2 + \frac{1}{2}\frac{\partial^2 S}{\partial x_j^2} + 2\frac{\partial S}{\partial t}. \tag{10}$$

By substituting (6) in (4) one obtains

$$\frac{\partial Z(x_0, 0; x, t)}{\partial t} = -\hat{H} Z(x_0, 0; x, t). \tag{11}$$

Equation (11) shows that $Z(x_0, 0; x, t)$ satisfies Bloch equation (2) with $\hat{H}$ the Hamiltonian given by (9) (Beilinson 1982, Beilinson 1979, Glim and Jaffe 1983). Finally, the function $\exp[2S(x, t)]$ in (6) satisfies the corresponding time-reversed Bloch equation

$$\hat{H}\,\mathrm{e}^{2S(x,t)} = \frac{\partial}{\partial t}(\mathrm{e}^{2S(x,t)}). \tag{12}$$

Therefore the stochastic equation (3) can be derived from a concrete solution of (12).

Since the potential energy given in equation (10) is at the lowest level, the presence of the potential $V(x)$ in the Hamiltonian makes the ground-state wavefunction vanish (Beilinson 1979, 1982, Glimm and Jaffe 1983). Consequently, for each Hamiltonian in Euclidean quantum mechanics there is a set of corresponding stochastic equations (3), which will depend on the solutions of (10) being considered. One should note that just one of the possible Fokker–Planck equations, namely the one corresponding to the ground-state wavefunction $\exp[2s(x)]$, admits a solution with a definite signal that can be interpreted as the density probability. Obviously

$$V(x)=2\frac{\partial S}{\partial \tau}+\left(\frac{\partial S}{\partial x_j}\right)^2+\frac{1}{2}\frac{\partial^2 S}{\partial x_j^2}=\left(\frac{\partial s}{\partial x_j}\right)^2+\frac{1}{2}\frac{\partial^2 s}{\partial x_j^2}. \tag{13}$$

Let us illustrate the above arguments with two examples from Euclidean problems:

(a) The hydrogen atom. The hydrogen atom ground-state wavefunction has the form $Z_0(r)=\exp(-2r)=\exp[2s(r)]$. Consequently, it follows that $s(r)=-r$. Substituting this expression into equation (13) one obtains $V(r)=-(2/r)+1$. Then Bloch equation results

$$\frac{\partial Z}{\partial t}=-\frac{1}{4}\frac{\partial^2 Z}{\partial r^2}+\left(1-\frac{2}{r}\right)Z \tag{14}$$

or, in terms of physical variables

$$\hbar\frac{\partial Z}{\partial(\mu/2\hbar)t}=-\frac{\hbar^2}{2\mu}\frac{\partial^2 Z}{\partial r^2}+\left(\frac{\mu\,\mathrm{e}^4}{2\hbar^2}-\frac{\mathrm{e}^2}{r}\right)Z. \tag{15}$$

Hence, we see that the Coulomb potential $-\mathrm{e}^2/r$ is automatically shifted upwards to the ground state level.

(b) The harmonic oscillator. For the harmonic oscillator, the ground-state wavefunction has the form $Z_0(x)=\exp(-\omega x^2)$. In this case $s(x)=-(\frac{1}{2})\omega x^2$ and the potential energy becomes $V(x)=\omega x^2-(\frac{1}{2})\omega$. Then Bloch equation results

$$\frac{\partial Z}{\partial t}=\frac{1}{4}\frac{\partial^2 Z}{\partial x^2}-(\omega x^2-\tfrac{1}{2}\omega)Z \tag{16}$$

or, in physical variables,

$$\hbar\frac{\partial Z}{\partial(m/2\hbar)t}=\frac{\hbar^2}{2m}\frac{\partial^2 Z}{\partial x^2}-\left(\frac{m\omega^2x^2}{2}-\tfrac{1}{2}\hbar\omega\right)Z. \tag{17}$$

We observe in this case that the potential energy of the harmonic oscillator is shifted downwards, to the ground state level.

The Langevine's stochastic equation is written, for the hydrogen atom case, as

$$\dot{y}_j+\frac{2}{r}y_j=\dot{\varphi}_j \qquad j=1,2,3 \tag{18}$$

and, for the harmonic oscillator case, as

$$\dot{x}_j+\omega x_j=\dot{\varphi}_j \qquad j=1,2,3. \tag{19}$$

If one considers that both equations have the same stochastic source $\dot{\varphi}_j$, which carries all time-spatial characteristics of the system, then it is possible to transform from one system to the other as follows

$$\dot{x}_j+\omega x_j=\dot{y}_j+\frac{2y_j}{r}. \tag{20}$$

In this sense, the one-to-one correspondence of the paths $x(\tau)$ and $y(\tau)$ can be used. Solving (20), one obtains

$$x_j(\tau)-x_j(0)\,\mathrm{e}^{-\omega t}=y_j(\tau)+\int_0^\tau \mathrm{e}^{-\omega(\tau-s)}\left[\frac{2}{r}y_j(s)-\omega y_j(s)\right]\mathrm{d}s. \tag{21}$$

On the other hand, the transition probability corresponding to the stochastic equation (19) is

$$W(x_0,0;x,t)=\int_{c^3}\exp\left[-\int_0^t(\dot{x}_j+\omega x_j)^2\,\mathrm{d}\tau+\tfrac{3}{2}\omega t\right]\prod_{\tau=0}^{t}\frac{\mathrm{d}x(\tau)}{(\sqrt{\pi\,\mathrm{d}\tau})^3}. \tag{22}$$

Considering that the Fredholm denominator in equation (21) is $D=\exp[-\frac{3}{2}\omega t+\int_0^t(1/r)\,\mathrm{d}\tau]$ the following expression is obtained for the transition probability

$$W(y_0,0;y,t)=\int_{c^3}\exp\left[-\int_0^t\left(\dot{y}_j+\frac{2}{r}y_j\right)^2\mathrm{d}\tau+\int_0^t\frac{1}{r}\,\mathrm{d}\tau\right]\prod_{\tau=0}^{t}\frac{\mathrm{d}y(\tau)}{(\sqrt{\pi\,\mathrm{d}\tau})^3}. \tag{23}$$

This is exactly the result that should be obtained by solving the system of stochastic equations when one uses the ground-state wavefunction of the hydrogen atom.

Thus we conclude that each problem of the Euclidean quantum mechanics can be obtained from any other problem, with the same number of degrees of freedom, through a substitution of functional variables. It should be pointed out that equation (3) provides the transition of a free particle (with coordinate φ) to a particle in the potential field (10) with coordinate x. It should still be noted that the strong interrelation between the Brownian motion problems and those of quantum mechanics allows a simplified numerical solution of concrete quantum mechanics problems. Instead of solving Schrödinger equations by traditional methods, a diffusion-type problem can be solved numerically in a considerably simpler manner with a further time-analytical extension of the obtained results. In other words, in Euclidean quantum mechanics (and, therefore, also in usual quantum mechanics) the quantum nature of the particles can be related, not with the particle itself, but with the stochastic spacetime background. As indicated above, the quantum phenomena are originated by this stochastic spacetime background, whose essence is the time derivative of the Wiener process. In this sense, each quantum particle is investigated, exactly, in this background interaction.

Let us recall the fundamental two-electron interaction problem in the He atom. It is known that the results of the quantum theory for this atom agree very well with experimental data. Being standard quantum particles, the electrons in the He atom are not yet sensitive to a specific quantum interaction through this stochastic background interaction, common to all quantum particles. That is why it should be considered that this background interaction has no correlation effect for distances of the Bohr radius order of magnitude (10^{-8} cm).

It should be pointed out that the nature of the usual quantum regularities for many particles probably resides on the lack of correlation of the background interaction in the particle sites. It can be thought that different quantum particles may be found, for a relatively long time, at distances considered shorter than the Bohr dimensions. This is the case, for instance, of the particles inside the atomic nucleus. Because they are in very small regions of the spacetime background interaction, there will be a strong correlation among them.

Let us examine the limit case in which this correlation is absolute. That is, let us examine the case of stochastic equations of the type

$$\dot{x}+\frac{\partial S}{\partial x}=\dot{\varphi} \tag{24}$$

and

$$\dot{y}+\frac{\partial R}{\partial y}=\dot{\varphi}. \tag{25}$$

In standard quantum mechanics, this case corresponds to the equations $\dot{x}+(\partial s/\partial x)=\dot{\varphi}_1$ and $\dot{y}+(\partial R/\partial y)=\dot{\varphi}_2$, where $\dot{\varphi}_1$ and $\dot{\varphi}_2$ are two different Wiener processes, as in equation (3). Equations (24) and (25) correspond to the same Wiener process $\varphi(\tau)$.

In order to simplify the problem, let us examine the case of two oscillators in the Euclidean quantum mechanics, located at the same point of the stochastic spacetime background

$$\dot{x}+\omega x=\dot{\varphi} \tag{26}$$

and

$$\dot{y}+\eta y=\dot{\varphi}. \tag{27}$$

Obviously, if the y coordinate is not considered (for instance, if it cannot be measured), then equation (27) disappears, leaving only (26), which is the equation that describes a harmonic oscillator in Euclidean quantum mechanics. This problem was considered previously. We are interested in defining the transition probability for the whole system (26) and (27), that is, for two harmonic oscillators, in the Euclidean formalism, located at the same point of the stochastic spacetime background. Besides the transition probability, we are also interested in determining the form of the path integral for this problem.

The solutions of the non-homogeneous equations (26) and (27) are, respectively:

$$x(\tau)-x_0\,\mathrm{e}^{-\omega t}=\int_0^{\tau}[2\delta(\tau-s)-\omega\,\mathrm{e}^{-\omega(\tau-s)}]\varphi(s)\,\mathrm{d}s \tag{28}$$

and

$$y(\tau)=y_0\,\mathrm{e}^{-\eta t}=\int_0^{\tau}[2\delta(\tau-s)-\eta\,\mathrm{e}^{-\eta(\tau-s)}]\varphi(s)\,\mathrm{d}s \tag{29}$$

that represents the conditions to be imposed on the linear functionals of $\varphi(\tau)$, with the kernels

$$a(\tau)=[2\delta(\tau-s)-\omega\exp[-\omega(\tau-s)]]$$

and

$$a'(\tau)=[2\delta(\tau-s)-\eta\exp[-\eta(\tau-s)]]$$

respectively. In equations (28) and (29), $\varphi(\tau)=\int_0^{\tau}2\delta(\tau-s)\varphi(s)\,\mathrm{d}s$ is the linear functional at the instant τ. The factor 2 is related to the integration limit (the integration variable s tends to the limit just by one side). Then, the probability density for these functionals is given by

$$W(x_0, y_0, 0; x, y, t)$$

$$= \int_c \delta\left\{(x - x_0\, e^{-\omega t}) - \int_0^t [2\delta(t-s) - \omega\, e^{-\omega(t-s)}]\varphi(s)\, ds\right\}$$

$$\times \left\{(y - y_0\, e^{-\eta t}) - \int_0^t [2\delta(t-s) - \eta\, e^{-\eta(t-s)}]\varphi(s)\, ds\right\} d_w\varphi(\tau). \qquad (30)$$

Writing the Fourier-transform of the δ-function and utilizing the Paley-Wiener formula for the linear functionals distribution, i.e.

$$\int_c \mathfrak{F}\left[\int_0^t a(\tau)\varphi(\tau)\, d\tau\right] d_w\varphi(\tau) = \int_{-\infty}^{+\infty} \mathfrak{F}(z)\frac{e^{-z^2/B}}{\sqrt{\pi B}} \qquad (31)$$

where $B = \int_0^t [\int_\tau^t a(s)\, ds]^2\, d\tau$, it can be shown (Gelfand and Vilenkin 1961, Yanovitch 1975) that:

$$W(x_0, y_0, 0; x, y, t)$$

$$= \exp \frac{\begin{vmatrix} 0 & x - x_0\, e^{-\omega t} & y - y_0\, e^{-\eta t} \\ x - x_0\, e^{-\omega t} & \dfrac{1 - e^{-2\omega t}}{2\omega} & \dfrac{1 - e^{-(\omega+\eta)t}}{(\omega+\eta)} \\ y - y_0\, e^{-\eta t} & \dfrac{1 - e^{-(\omega+\eta)t}}{(\omega+\eta)} & \dfrac{1 - e^{-2\eta t}}{2\eta} \end{vmatrix}}{(\det a_{jk})}$$

$$\times (\pi^2 (\det a_{jk}))^{-1/2} \qquad (32)$$

where

$$(\det a_{jk}) = \begin{vmatrix} \dfrac{1 - e^{-2\omega t}}{2\omega} & \dfrac{1 - e^{-(\omega+\eta)t}}{(\omega+\eta)} \\ \dfrac{1 - e^{-(\omega+\eta)t}}{(\omega+\eta)} & \dfrac{1 - e^{-2\eta t}}{2\eta} \end{vmatrix}. \qquad (33)$$

Equation (31) above must represent a continuous bounded functional of continuous functions in $\mathbb{C}$ (Yanovitch 1975).

So, a Gaussian distribution is obtained for the quantities $x - x_0\, e^{-\omega t}$ and $y - y_0\, e^{-\eta t}$, with the inverse matrix a_{jk}. In the limiting case of equal frequencies $\omega = \eta$ it turns out that $W(x_0, y_0, 0; x, y, t) = \delta[x - x_0 - (y - y_0)]$, i.e. the processes $x(\tau)$ and $y(\tau)$ coincide thus leading to an absolute (kinematical) relationship in the considered example.

In the following, the path integral for our problem will be obtained. Starting with the power-series expansion of $(\det a_{jk})$:

$$(\det a_{jk}) = (\det a_{jk})_{t=0} + \frac{1}{1!}(\det a_{jk})'_{t=0}t + \frac{1}{2!}(\det a_{jk})''_{t=0}t^2 + \dots$$

$$\simeq \tfrac{1}{12}(\omega - \eta)^2 t^4 - \tfrac{1}{12}(\omega^2 + \eta^2)(\omega + \eta)t^5 + \mathcal{D}(t^6) \qquad (34)$$

as well as of the numerator of the exponent in (32):

$$\begin{vmatrix} 0 & x-x_0\,\mathrm{e}^{-\omega t} & y-y_0\,\mathrm{e}^{-\eta t} \\ x-x_0\,\mathrm{e}^{-\omega t} & \dfrac{1-\mathrm{e}^{-2\omega t}}{2\omega} & \dfrac{1-\mathrm{e}^{-(\omega+\eta)t}}{(\omega+\eta)} \\ y-y_0\,\mathrm{e}^{-\eta t} & \dfrac{1-\mathrm{e}^{-(\omega+\eta)t}}{(\omega+\eta)} & \dfrac{1-\mathrm{e}^{-2\eta t}}{2\eta} \end{vmatrix}$$

$$\simeq [(\dot{x}+\omega x)-(\dot{y}+\eta y)]t^3-\tfrac{1}{12}[(\ddot{x}-\omega^2 x)-(\ddot{y}-\eta^2 y)]t^5+\mathcal{O}(t^6) \tag{35}$$

where the identity

$$(\dot{x}+\omega x)(\eta-\omega)=(\ddot{x}-\omega^2 x)-(\ddot{y}-\eta^2 y)$$

was used. Taking the limit $t\to 0$ in the expression for $W(x_0, y_0, 0; x, y, t)$ yields

$$W(x_0, y_0, 0; x, y, t) \xrightarrow[t\to 0]{} \delta\{(\dot{x}+\omega x)-(y+\dot{\eta} y)\}\exp\{-[(\ddot{x}-\omega^2 x)-(\ddot{y}-\eta^2 y)]^2 t\}. \tag{36}$$

From this result, the obtained expression for $W(x_0, y_0, 0; x, y, t)$ in terms of the path integral is

$$W(x_0, y_0, 0; x, y, t) = \int \delta\{(\dot{x}+\omega x)-(\dot{y}+\eta y)\}\exp\left\{-\int_0^t [(\ddot{x}-\omega^2 x)-(\ddot{y}-\eta^2 y]^2\right\}\mathrm{d}\tau \times \prod_{\tau=0}^{t}\frac{\mathrm{d}x(\tau)\,\mathrm{d}y(\tau)}{(\sqrt{\pi\,\mathrm{d}\tau})^2}. \tag{37}$$

The main reason to rewrite $W(x_0, y_0, 0; x, y, t)$ in the form of equation (37) is that in this form the causal nature of transition probability density becomes clear since (32) defines a probabilistic measure corresponding to a cylindric set of finite-measure basis in the two-dimensional vector-function space $\{x(\tau), y(\tau)\}$. This causality condition is verified through the Einstein-Smolukhovsky equation

$$W(x_0, y_0, 0; x, y, t) = \int_{-\infty}^{\infty} W(x_0, y_0, 0; x_\tau, y_\tau, \tau)\, W(x_\tau, y_\tau, \tau; x, y, t)\,\mathrm{d}x_\tau\,\mathrm{d}y_\tau. \tag{38}$$

The result (37) reflects a non-trivial type of interrelation between the processes x and y through a unique stochastic background interaction. Probably such an interaction may play an imprtant role in the description of nucleons.

The specific interaction of quantum particles considered here, which takes place whenever particles stay close ehough to one another for a sufficiently long time interval (a realization of such a situation probably being quarks confined into an atomic nucleus), can be illustrated by the interaction between two quantum oscillators with a potential energy

$$V=\tfrac{1}{2}\omega x^2-\tfrac{1}{2}\omega+\tfrac{1}{2}\eta^2 x^2-\tfrac{1}{2}\eta \qquad (m=\hbar=1).$$

Conversely, in the usual quantum theory, these oscillators, in general, do not interact. Under the conditions assumed in this work, an interaction arises between the oscillators that is due to the stochastic spacetime background (see equation (32)). In this way, x and y are not to be considered as normal coordinates, thus leading to a shifting of their normal frequencies.

Acknowledgments

EPL acknowledges financial support by Patrice Lumumba People's Friendship University (UPL). EPL would like to thank the Department of Theoretical Physics of this university, especially the head of this department, Professor Y P Terletsky, for the kind hospitality during his visit.

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